On a Possibility of Membrane Cosmology

S.N. Roshchupkin

Simferopol State University

333036, Simferopol, Ukraine

and A.A. Zheltukhin*

Kharkov Institute of Physics and Technology

310108, Kharkov, Ukraine

Tensionless null p-branes in arbitrary cosmological backgrounds are considered and their motion equations are solved. It is shown that an ideal fluid of null p-branes may be considered as a source of gravity for D-dimensional Friedmann-Robertson-Walker universes.

The string approach [1] to building a selfconsistent inflation theory has evoked a great deal of interest now. Therefore it is important to search the string dynamics in curved space-time [1-6]. Clearing up a physical role of p-branes in cosmology is a topical problem, too [4,5]. Here we consider dynamics of null p-branes [7,8], which are tensionless limit of p-branes, in curved spaces. We show that their motion equations can be linearized and exactly solved in contrast to the case of p-branes. Generalizing the results of [5] we find that the perfect fluid of the null p-branes is an alternative dominant source of gravity in the Hilbert-Einstein equations for the Friedmann universe with $k = 0$. The action for null p–branes in a cosmological background $G_{MN}(x)$ may be written as [7]

$$S = \int d^{p+1}\xi \, \frac{det(\partial_\mu x^M \, G_{MN}(x) \, \partial_\nu x^N)}{E(\tau, \sigma^n)}, \tag{1}$$

*E-mail:kfti@kfti.kharkov.ua

where $\mu, \nu = 0, 1, ..., p$ are the indices of the hyperworldsheet of the null p-brane ($\xi^0 = \tau, \xi^1 = \sigma^1, \xi^2 = \sigma^2, ..., \xi^p = \sigma^p$) and $E(\tau, \sigma^n)$ is a $(p+1)$ –dimensional hyperworldsheet density. The determinant g of the induced null p–brane metric $g_{\mu\nu}$

$$g_{\mu\nu} = \partial_\mu x^M \; G_{MN}(x) \; \partial_\nu x^N = \begin{pmatrix} \dot{x}^A \; G_{AB}(x) \; \dot{x}^B & \dot{x}^A \; G_{AB}(x) \; \partial_n x^B \\ \partial_m x^A \; G_{AB}(x) \; \dot{x}^B & \hat{g}_{mn}(x) \end{pmatrix},$$

$$\hat{g}_{mn}(x) = \partial_m x^A \; G_{AB}(x) \; \partial_n x^B \tag{2}$$

may be presented in a factorized form

$$g = \dot{x}^M \; \tilde{\Pi}_{MN}(x) \; \dot{x}^N \; \hat{g},$$

$$\hat{g} = \det \hat{g}_{mn}, \tag{3}$$

where point denotes differentiation with respect to τ . The matrix $\tilde{\Pi}_{MN}(x)$ has the properties of the projection operator [8]

$$\tilde{\Pi}_{MN} = G_{MN} - G_{MB} \; \partial_m x^B \; \hat{g}^{-1mn} \; \partial_n x^L \; G_{LN} \tag{4}$$

Therefore the action(1)can be written in the following form:

$$S = \int d^{p+1}\xi \; \frac{\dot{x}^M \; \tilde{\Pi}_{MN}(x) \; \dot{x}^N \; \hat{g}}{E(\tau, \sigma^n)}, \tag{5}$$

The variation of the action S with respect to E generates the degeneracy condition for the induced metric $g_{\mu\nu}$

$$g = \det g_{\mu\nu} = 0, \tag{6}$$

which separates the class of $(p+1)$ –dimensional isotropic geodesic hypersurfaces characterized by the null volume. In the gauge

$$\dot{x}^M G_{MN} \partial_m x^N = 0; \qquad \left(\frac{\hat{g}}{E}\right)^{\bullet} = 0 \tag{7}$$

we find the motion equations and constraints in the following form

$$\ddot{x}^M + \Gamma^M_{PQ}\ \dot{x}^P \dot{x}^Q = 0$$

$$\dot{x}^M G_{MN} \dot{x}^N = 0, \qquad \dot{x}^M G_{MN} \partial_m x^N = 0 \tag{8}$$

Now consider the case of the D –dimensional Friedmann universe with $k = 0$ described by the metric form

$$ds^2 = G_{MN} dx^M dx^N = (dx^0)^2 - R^2(x^0)\ dx^i \delta_{ik} dx^k, \tag{9}$$

where $M, N = 0, 1, ..., D-1$. It is convenient to transform Eqs.(8)to the conformal time $\tilde{x}^0(\tau, \sigma)$, defined by

$$dx^0 = C(\tilde{x}^0) d\tilde{x}^0, \qquad C(\tilde{x}^0) = R(x^0), \qquad \tilde{x}^i = x^i \tag{10}$$

In the gauge of the conformal time the metric (9) is presented in the conformal-flat form

$$ds^2 = C(\tilde{x}^0) \eta_{MN} d\tilde{x}^M d\tilde{x}^N, \qquad \eta_{MN} = diag(1, -\delta_{ij}) \tag{11}$$

with the Christoffel symbols $\tilde{\Gamma}^M_{PQ}(\tilde{x})$ [9]

$$\tilde{\Gamma}^M_{PQ}(\tilde{x}) = C^{-1}(\tilde{x})[\delta^M_P \tilde{\partial}_Q C + \delta^M_Q \tilde{\partial}_P C - \eta_{PQ} \tilde{\partial}^M C] \tag{12}$$

Taking into account the relations (10,12) we transform Eqs. (8) to the form

$$\ddot{\tilde{x}}^M + 2C^{-1} \dot{C} \dot{\tilde{x}}^M = 0 \tag{13}$$

$$\eta_{MN} \dot{\tilde{x}}^M \dot{\tilde{x}}^N = 0, \qquad \eta_{MN} \dot{\tilde{x}}^M \partial_m \tilde{x}^N = 0, \tag{14}$$

where $m, n = 1, ..., p$. The first integration of these equations leads to the following equations of the first order

$$H^* C^2 \dot{\tilde{x}}^0 = \psi^0(\sigma^1, \sigma^2, .., \sigma^p), \qquad H^* C^2 \dot{\tilde{x}}^i = \psi^i(\sigma^1, \sigma^2, .., \sigma^p), \tag{15}$$

the solutions of which have the form

$$\tau = H^* \psi_0^{-1} \int_{t_o}^{t} dt\ R(t), \tag{16}$$

$$x^i(\tau, \sigma^1, \sigma^2, .., \sigma^p)) = H^{*-1}\psi^i \int_0^\tau d\tau \ R^{-2}(t),$$

where H^* is a metric constant with the dimension L^{-1} and $t_0 \equiv x^0(0, \sigma^1, \sigma^2, .., \sigma^p)$, and $x^i(0, \sigma^1, \sigma^2, .., \sigma^p))$ and $\psi^M(\sigma)$ are the initial data. The solution (16) for the space world coordinates $x^i(t)(i = 1, ..., D-1)$ as a function of the cosmic time $t = x^0$, may be written in the equivalent form as

$$x^i(t, \sigma^1, \sigma^2, .., \sigma^p) = x^i(t_0, \sigma^1, \sigma^2, .., \sigma^p) + \nu^i(\sigma^1, \sigma^2, .., \sigma^p) \int_{t_o}^t \ dt \ R^{-1}(t), \tag{17}$$

where $\nu^i(\sigma) \equiv \psi_0^{-1}\psi^i$. The explicit form of the solutions (16) allows to transform the constraints (11) into those for the Cauchy initial data:

$$\nu^i(\sigma^m)\nu^k(\sigma^n)\delta_{ik} = 1, \tag{18}$$

$$\partial_m x^0(0, \sigma^m) = R(x^0(0, \sigma^m) \ \nu^i(\sigma^n)\delta_{ik}\partial_m x^k(0, \sigma^m), \tag{19}$$

where $\nu^k(\sigma^1, \sigma^2, .., \sigma^p) = \psi^i\psi_0^{-1}$. Note that the constraints (18) and (19) produce the additional constraints , which are their integrability conditions

$$\partial_m(\nu_i(\sigma^n)\partial_n x^i(0, \sigma^l)) - \partial_n(\nu_i(\sigma^n)\partial_m x^i(0, \sigma^l)) = 0 \tag{20}$$

Now we want to show that the null p-branes may be considered as dominant gravity sources of the Friedmann universes. With this aim assume that the perfect fluid of these null p-branes is homogenious and isotropic. The energy density $\rho(t)$ and the pressure $p(t)$ of this fluid and its energy -momentum $\langle T_{MN} \rangle$ are connected by the standard relations

$$\langle T_0{}^0 \rangle = \rho(t), \qquad \langle T_i{}^j \rangle = -p(t)\delta_i{}^j = -\frac{\delta_i{}^j}{D-1} \ \frac{A}{R^D(t)}, \tag{21}$$

The tensor $\langle T_{MN} \rangle$ is derived from the momentum–energy tensor T_{MN} of a null p–brane by means of its space averaging when a set of null p–branes is introduced instead of a separate null p–brane. The energy -momentum tensor $T^{MN}(x)$ of null p–brane is defined by the variation of the action (1) with respect to $G_{MN}(x)$

$$T^{MN}(X) = \frac{1}{\pi\gamma^* \sqrt{|G|}} \int \ d\tau d^p\xi \ \dot{x}^M \dot{x}^N \delta^D(X^M - x^M) \tag{22}$$

After the substitution of the velocities $\tilde{x}^m$ (15) and subsequent integration with respect to τ , the non-zero components of T_{MN} (22) take the following form

$$T^{00}(X) = \frac{1}{\pi\gamma^* H^*} R^{(-D)}(t) \int d\tau d^p\xi\ \psi_0(\sigma^m)\delta^{D-1}(X^i - x^i(\tau, \sigma^m)),$$

$$T^{ik}(X) = \frac{1}{\pi\gamma^* H^*} R^{(-D-2)} \int d\tau d^p\xi\ \nu^i(\sigma)\nu^k(\sigma)\psi_0(\sigma)\delta^{D-1}(X^i - x^i(\tau, \sigma^m), \qquad (23)$$

where the time dependence T^{MN} is factorized and accumulated in the scale factor $R(t)$. Taking into account the constraint $\nu^i(\sigma^m)\nu^k(\sigma^n)\delta_{ik} = 1$ gives rise to the following relation between the components of the tensor $\langle T_{MN}\rangle$

$$Sp\ T = T_0{}^0 + G_{ij}T^{ij} = 0. \qquad (24)$$

As a result of the space averaging we find the non–zero components of $\langle T_{MN}\rangle$ to be equal to

$$\langle T_0{}^0\rangle = \rho(t) = \frac{A}{R^D(t)}, \qquad \langle T_i{}^j\rangle = -p(t)\delta_i{}^j, \qquad (25)$$

where A is a constant with the dimension L^{-D} . Eqs.(25) show that the equation of state of null p–branes fluid is just the equation of state for a gas of massless particles

$$\langle Sp\ T\rangle = \langle T_M{}^M\rangle = 0 \Longleftrightarrow \rho = (D-1)p \qquad (26)$$

Eq. (26) was found in [3]as the approximate equation of state describing the phase of perfect gas of shrunk strings $(R(t) \to 0,\ \tau \to 0)$ valid in the small $R(t)$ limit for a negatively accelerated contraction $(d^2R/dt^2 < 0,\ dR/dt < 0)$ of the universes. Taking into account the additional interactions of the null p–brane fluid with any background fields breaks the coincidence of their state equation and the one describing ultrarelativistic gas of massless particles in the same background. It is a consequence of the fact that particles have no internal structure in contrast to extended objects such as p–branes. An example of such an interaction may be the interaction of null p–brane with the dilaton field ϕ describing by the action [7]

$$S = S_0 + S_1 = \int d\tau d^p\xi \left[\frac{det(\partial_\mu x^M G_{MN}\partial_\nu x^N)}{E(\tau, \sigma^m)} - \phi(x^M)E(\tau, \sigma^m)\right] \ , \qquad (27)$$

Now assume that the fluid of null p–branes is a dominant source of the FRW gravity (1). For the validity of the last conjecture it is necessary that the H–E equations

$$R_M{}^N = 8\pi G_D\langle T_M{}^N\rangle \qquad (28)$$

with the non-zero Ricci tensor $R_M{}^N$ components defined by $G_M{}^N$ (1)

$$R_0{}^0 = -\frac{D-1}{R}\frac{d^2R}{dt^2},$$

$$R_i{}^k = -\delta_i{}^k\left[\frac{1}{R}\frac{d^2R}{dt^2} + \frac{D-2}{R^2}\left(\frac{dR}{dt}\right)^2\right] \tag{29}$$

should contain the tensor $\langle T_M{}^N\rangle$ (21) as a source of the FRW gravity. Moreover, the constraints (21), i.e.

$$\rho R^D - A = 0$$

must be a motion integral for the HE system (28). It is actually realized because

$$\frac{d}{dt}(\rho R^D) = -\frac{D-2}{16\pi G_D}R^{D-1}\frac{dR}{dt}R_M{}^M = 0,$$

since the trace $R_M{}^M \sim \langle T_M{}^M\rangle = 0$ (see (26). In view of this fact it is enough to consider only one equation of the system (28)

$$\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = \frac{16\pi G_D}{(D-1)(D-2)}\frac{A}{R^D} \tag{30}$$

which defines the scale factor R(t) of the FRW metric (1). Note that in the case $D = 4$ Eq.(30) transforms into the well-known Friedmann equation for the energy density in the radiation dominated universe with $k = 0$. The solutions of Eq. (30) are

$$R_I(t) = [q(t_c - t)]^{2/D}, \qquad t < t_c,$$

$$R_{II}(t) = [q(t - t_c)]^{2/D}, \qquad t > t_c,$$

where $q = [4\pi G_D A/(D-1)(D-2)]^{1/2}$ and t_c is a constant of integration which is a singular point of the metric. The solution R_I describes the stage of negatively accelerated contraction of D-dimensional FRW universe. In the small R limit ($R \to 0$) [3] this solution was found as an approximate asymptotic solution for the gas of strings with non-zero tension. For the case of null strings [5] and null p-branes this solution is exact. The second solution (30) R_{II} describes the stage of the negatively accelerated expansion of the FRW universe from the state with space volume equal to "zero". Thus we see that the perfect fluid of noninteracting null p-branes may be considered as an alternative source of the gravity in the FRW universes with $k = 0$.

Null p–branes theory in curved space–time is characterized by a set of the constraints connected with the reparametrization symmetry. To find these constraints consider the canonical momentum of null p–brane $\mathcal{P}_M$ conjugated to its world coordinate x_M

$$\mathcal{P}_M = 2E^{-1}\,\hat{g}\,\tilde{\Pi}_{MN}(x)\,\dot{x}^N \tag{31}$$

Then we find the following primary constraints

$$G_{MN}\partial_m x^N \mathcal{P}^M = 0 \tag{32}$$

The Hamiltonian density produced by the action functional (1) is

$$\mathcal{H}_0 = \frac{1}{4}E\,\hat{g}^{-1}G^{MN}\mathcal{P}_M\mathcal{P}_N \tag{33}$$

and the condition of conservation of the primary constraint

$$\mathcal{P}_{(E)} = 0, \tag{34}$$

where $\mathcal{P}_{(E)}$ is the canonical momentum conjugated to E, generates the following condition

$$\dot{\mathcal{P}}_{(E)} = \int d^p\xi\ \{\mathcal{H}_0, \mathcal{P}_{(E)}\}_{P.B.} = -\frac{1}{4\hat{g}}G^{MN}\mathcal{P}_M\mathcal{P}_N = 0 \tag{35}$$

produces a secondary constraint

$$G^{MN}\mathcal{P}_M\mathcal{P}_N = 0 \tag{36}$$

Additional constraints do not appear so the total hamiltonian of null p-brane is given by

$$H = \int d^p\xi\ [\lambda^m(G_{MN}\partial_m x^N \mathcal{P}^M) + \frac{E}{4\hat{g}}G^{MN}\mathcal{P}_M\mathcal{P}_N + \omega\mathcal{P}_{(E)}], \tag{37}$$

This hamiltonian and the reparametrization constraints may be used for the quantization of null p–brane in a curved space time.

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